An overview of the Breuil-Schneider conjecture

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Let $K/\mathbb{Q}_p$ be a finite extension.

$$\Gamma_K = \mathrm{Gal}(\overline{K}/K) = \text{absolute Galois group}$$

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$$W_K = \text{Weil group}$$

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These groups fit in the diagram

$$\begin{CD}
0 @>>> I_K @>>> \Gamma_K @>>> \widehat{\mathbb{Z}} @>>> 0 \\
@. @| @AAA @AA{\text{dense}}A @. \\
0 @>>> I_K @>>> W_K @>>> \mathbb{Z} @>>> 0.
\end{CD}$$

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⋆ **Example**. A/K an abelian variety, $g = \dim(A)$. The Tate module

$$T_\ell A = \varprojlim_r A[\ell^r]$$

carries a Γ_K-action. Gives a $2g$-dimensional representation $V_\ell A = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell A$.

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– For any Galois representation ρ as above,

$$\rho \rightsquigarrow \pi_{\text{sm}}(\rho) = \text{a } \textit{smooth} \text{ representation of } \mathrm{GL}_n(K).$$

This is essentially the local Langlands correspondence.

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(Here $\phi \in W_K$ is a lift of Frobenius, and $t_\ell : I_K \twoheadrightarrow \mathbb{Z}_\ell$.)

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The choice of Λ gives a $\mathrm{GL}_n(K)$-invariant *norm* $\|\cdot\|$ on $\pi_{\mathrm{sm}}(\rho)$,

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$\star$ **Breuil-Schneider**. What's the story for $\ell = p$?

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Then, we combine them into a *locally algebraic* representation:

$$\mathrm{BS}(\rho) = \pi_{\mathrm{alg}}(\rho) \otimes \pi_{\mathrm{sm}}(\rho).$$

(A $\overline{\mathbb{Q}}_p$-vector space, with $\mathrm{GL}_n(K)$ acting diagonally.)

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+ local-global compatibility $\rightsquigarrow$ Fontaine-Mazur conjecture (for odd ρ).

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This is an n-dimensional $\overline{\mathbb{Q}}_p$-vector space with "linear algebra data"

$$(\phi, N, \mathrm{Fil}^i D).$$

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(Here t_N depends only on ϕ, whereas t_H depends only on the filtration.)

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Saying ρ is *regular* means they are distinct – in other words

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The Hodge-Tate weights give a tuple

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) := -(i_n, i_{n-1}, \ldots, i_1) - (0, 1, \ldots, n-1).$$

This is a *dominant* weight for GL_n. (I.e., $a_1 \leq a_2 \leq \cdots \leq a_n$.)

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$\pi_{\text{alg}}(\rho)$ = irreducible algebraic rep of GL_n with highest weight $\mathbf{a}$.

$\mathrm{WD}(\rho) = (r, N)$ is a Weil-Deligne representation on D,

- $r(w) = \phi^{-d(w)}$ where $d : W_K \twoheadrightarrow \mathbb{Z}$,
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(a generic representation, i.e. $\exists$ Whittaker model, but possibly reducible).

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The implication $\Uparrow$ *holds.*

The other direction $\Downarrow$ remains open in general.

In fact, Hu proved a *stronger* statement:

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$\pi_{\mathrm{alg}}(\mathrm{HT}) \otimes \pi_{\mathrm{sm}}(\mathrm{WD})$ satisfies the *Emerton condition*.

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– What's the Emerton condition? W = locally algebraic rep of $\text{GL}_n(K)$.

$$W^{N_0, Z_M^+ = \chi} \neq 0 \implies |\delta_P(z)^{-1}\chi(z)| \leq 1, \ \forall z \in Z_M^+.$$

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$\rightsquigarrow$ a group-theoretic formulation of the *admissibility* of $\{\text{Fil}^i D\}_{i \in \mathbb{Z}}$.

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This happens if

$$\pi_{\mathrm{sm}}(\mathrm{WD}) = Q(\Delta) \otimes |\det|^{\frac{1-n}{2}}$$

is a *generalized Steinberg* representation. ($\Longleftrightarrow$ WD is indecomposable.)

Notation:

- $n = \underbrace{m + \cdots + m}_{r}, \quad P_m = M_m N_m$ parabolic in GL_n,
- σ = supercuspidal representation of $\mathrm{GL}_m(K)$
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⋆ **Example**. ($m = 1$) Here $Q(\Delta)$ is a twist of the *Steinberg* representation;

$$\{\text{smooth functions on } B\backslash G\} \twoheadrightarrow \mathrm{St}_G.$$

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Theorem (S.)

The Breuil-Schneider conjecture holds when WD is indecomposable.

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This means:

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– The supercuspidal case was known (easy). The Steinberg case was new.

I proved a more general version for *any* connected reductive $G/\mathbb{Q}_p$.

- ξ = irreducible algebraic representation of G (over $\overline{\mathbb{Q}}_p$)
- π = essentially *discrete series* representation of $G(\mathbb{Q}_p)$

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– The norms come from *automorphic forms* on a model $\mathcal{G}/\mathbb{Q}$ such that

- $\mathcal{G}(\mathbb{R})$ is compact,
- $\mathcal{G}(\mathbb{Q}_p) = G(\mathbb{Q}_p)$.

(Such $\mathcal{G}$ exist by Borel-Harder. Think of unitary groups in the GL_n-case.)

Automorphic forms on $\mathcal{G}$,

$$A(\mathcal{G}) = \{\text{smooth functions } \underbrace{\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})}_{\text{compact}} \xrightarrow{f} \mathbb{C}\}.$$

Pick an $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ and identify ξ with a rep of $\mathcal{G}(\mathbb{C}) \supset \mathcal{G}(\mathbb{R})$. Call it $\xi_{\mathbb{C}}$.

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– Consider the multiplicity space

$$\mathrm{Hom}_{\mathcal{G}(\mathbb{R})}\big(\xi_{\mathbb{C}}, A(\mathcal{G})\big) = \bigoplus_{\Pi:\Pi_\infty \simeq \xi_{\mathbb{C}}} m(\Pi)\Pi_{\mathrm{fin}}$$

($\Pi = \Pi_\infty \otimes \Pi_{\mathrm{fin}}$ runs over the *automorphic representations* of $\mathcal{G}(\mathbb{A})$ of weight $\xi_{\mathbb{C}}$.)

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Exercise

As a $\mathcal{G}(\mathbb{A}_{fin})$-representation,

$$Hom_{\mathcal{G}(\mathbb{R})}\big(\xi_{\mathbb{C}}, A(\mathcal{G})\big) \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \{F : \mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}_{fin}) \longrightarrow \xi^\vee\}^{sm},$$

where $(gF)(x) := g_p \cdot F(xg)$ for $g \in \mathcal{G}(\mathbb{A}_{fin})$.

This gives $\mathcal{G}(\mathbb{A}_{\mathrm{fin}})$-equivariant embeddings,

$$\xi \otimes \left(\Pi_{\mathrm{fin}} \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p\right) \hookrightarrow \xi \otimes \left(\mathrm{Hom}_{\mathcal{G}(\mathbb{R})}\big(\xi_{\mathbb{C}}, A(\mathcal{G})\big) \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p\right)$$
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The sup-norm $\|\varphi\| := sup_{x\in\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}_{fin})}|\varphi(x)|_{\overline{\mathbb{Q}}_p}$ *is a* $\mathcal{G}(\mathbb{A}_{fin})$*-invariant norm.*

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$\rightsquigarrow$ If $\pi \simeq \Pi_p \otimes_{\mathbb{C},\iota} \overline{\mathbb{Q}}_p$, for an automorphic Π of weight $\xi_{\mathbb{C}}$ as above, then

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The existence of Π follows from standard *trace formula* methods:

Theorem (Bernstein, Clozel, Deligne, Kazhdan, ...)

Let S be a finite set of places, and let

$$\{\pi_v\}_{v \in S} \text{ be any collection of } \underline{\text{discrete}}\ \underline{\text{series}} \text{ representations of } \mathcal{G}(\mathbb{Q}_v).$$

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– The key point is π_v has a *pseudo-coefficient*; a function f_v on $\mathcal{G}(\mathbb{Q}_v)$ s.t.

$$\mathrm{tr}\sigma_v(f_v) = \begin{cases} 1 & \text{if } \sigma_v \simeq \pi_v \\ 0 & \text{if } \sigma_v \not\simeq \pi_v \text{ (and } \sigma_v \text{ is tempered).} \end{cases}$$

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$\star$ **Application**. Take $S = \{\infty, p\}$, $\pi_\infty = \xi_{\mathbb{C}}$, $\pi_p = \pi_{\mathbb{C}}$. ✓

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Taylor-Wiles patching $\rightsquigarrow$

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From modules of automorphic forms much like $A(\mathcal{G})$ they construct

$$M_\infty \quad - \quad \text{a module over } R_\infty = R_{\bar{\rho}}^{\square}[\![x_1, \ldots, x_N]\!] \text{ with } \mathrm{GL}_n(K)\text{-action.}$$

Using this construction they show:

Theorem (CEGGPS)

Assume $p \nmid 2n$. Let $\rho : \Gamma_K \to GL_n(\overline{\mathbb{Q}}_p)$ be potentially crystalline of regular weight s.t.

- *ρ is generic (i.e., $\pi_{sm}(\rho)$ is given by local Langlands);*

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What's an "automorphic component"? $\mathrm{WD}(\rho)$ gives an inertial type $\tau := r|_{I_K}$.

$$\rightsquigarrow \sigma = \sigma_{\mathrm{sm}} \otimes \sigma_{\mathrm{alg}} = \text{a locally algebraic rep of } \mathrm{GL}_n(\mathcal{O}_K) \text{ over } \overline{\mathbb{Q}}_p.$$

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– By local-global compatibility "at p" there's a map $R^{\square}_{\bar{\rho}}(\sigma) \twoheadrightarrow R_{\infty}(\sigma)$, and

$$\operatorname{Spec} R_{\infty}(\sigma)[1/p] \subseteq \operatorname{Spec} R^{\square}_{\bar{\rho}}(\sigma)[1/p]$$

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⋆ **Folklore**. All components are expected to be automorphic.

Danke schön.