Equivariant line bundles on the Drinfeld Tower and p-adic local Langlands correspondence (part 2)

Integral line bundles and $\mathscr{O}(-1)$

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Notations

Let $K/\mathbb{Q}_p$ finite, $\mathcal{O}_K$, ϖ, $\mathbb{F} = \mathbb{F}_q = \mathcal{O}_K/\varpi$ and $\breve{K} = \hat{K}^{ur}$.

1. $\mathbb{H}$ the Drinfeld upper half-plane over K: $\mathbb{H}(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(K)$,
2. $\hat{\mathbb{H}}$ the canonical semi-stable model and $\overline{\mathbb{H}}$ its special fiber,
3. $(\Sigma^n)_n$ the Drinfeld tower,
4. $G = \mathrm{GL}_2(K)$ and $G^+ \subset G$: $g \in G$ s.t. $v(\det(g)) \in 2\mathbb{Z}$ (index 2),
5. $\mathscr{O}(k) =$ restriction to $\mathbb{H}$ of the natural sheaf on $\mathbb{P}^1$ or its pullback to Σ^n.

Our goal today is to study $\mathrm{Pic}_{G^+}(\hat{\mathbb{H}})$ the group of G^+-equivariant line bundles on $\hat{\mathbb{H}}$.

The result

Here $K = \mathbb{Q}_p$. Last time, we have completely described $\mathscr{O}_{\Sigma^n}$ and $\mathscr{O}_{\Sigma^n}(-2)$ and have shown how to construct the representations $\Pi(V)^{an}$ for V de Rham of Hodge-Tate weights $(0,1)$ of discrete series type inside the de Rham complex of the covers.

Theorem (J.)

There is a subclass $\mathrm{Pic}_{G^+}^{w=-1,\geq 0}(\hat{\mathbb{H}}) \subset \mathrm{Pic}_{G^+}(\hat{\mathbb{H}})$ of "positive line bundles of weight -1" for which the map

$$\mathcal{L} \mapsto (\mathcal{L}(\hat{\mathbb{H}})/p)^* = \overline{\mathcal{L}}(\overline{\mathbb{H}})^*$$

gives a bijection between

1. $\mathrm{Pic}_{G^+}^{w=-1,\geq 0}(\hat{\mathbb{H}})/\sim$ (up to characters trivial mod p)
2. supersingular mod p representations of G^+.

One step beyond

Corollary

For any $\mathcal{L} \in \mathrm{Pic}_{G^+}^{w=-1,\geq 0}(\hat{\mathbb{H}})$, $\mathcal{L}(\hat{\mathbb{H}})[1/p]^*$ is an irreducible Banach space representation.

Conjecture

The map

$$\mathcal{L} \mapsto \mathrm{ind}_{G^+}^{G} \mathcal{L}(\hat{\mathbb{H}})[1/p]^*$$

gives a bijection between

1. $\mathrm{Pic}_{G^+}^{w=-1,\geq 0}(\hat{\mathbb{H}})/\{\mathcal{L} \sim \mathcal{L}^g : g \notin G^+\}$,
2. Banach space representations $\Pi(V)$ for V de Rham of Hodge-Tate weights $(0, 0)$.

Historic : Generic case

1. Morita and Murase : $\mathcal{O}(2k)$ on $\mathbb{H}$ (dimension 1),
2. Schneider-Stuhler (Pohlkamp) : More G-equivariant vector bundles in any dimension,
3. Orlik (Orlik-Strauch, Linden in char p) : Any G-equivariant vector bundles that are restrictions from $\mathbb{P}^d$.

Historic : Integral case

1. Teitelbaum : G-equivariant integral structure for $\mathscr{O}(2k)$ (dimension 1),
2. Grosse-Klönne : G-equivariant integral structure for $\mathscr{O}(k)$ with pathologies at singularities for odd k (dimension 1)
3. Grosse-Klönne (again) : Extension to some line bundles in higher dimension.

Philosophy

To avoid these problems, study G^+-equivariant bundles instead of G-equivariant bundles.

Integral structures and representations

Given $\mathcal{L}$ on $\hat{\mathbb{H}}$ we get $\mathcal{L}[1/p]$ on $\mathbb{H}$ and $\overline{\mathcal{L}}$ on $\overline{\mathbb{H}}$ and also four representations

$(\mathrm{H}^i(\hat{\mathbb{H}}, \mathcal{L})[1/p])^*$ $(i = 0, 1)$: Banach space representation	$\mathrm{H}^0(\mathbb{H}, \mathcal{L}[1/p])^*$: Locally analytic representation
$(\mathrm{H}^i(\hat{\mathbb{H}}, \mathcal{L})/p)^*$ $(i = 0, 1)$: Smooth mod p representation	$(\mathrm{H}^i(\overline{\mathbb{H}}, \overline{\mathcal{L}}))^*$ $(i = 0, 1)$: Smooth mod p representation

Link with modular forms

Last time : some quaternionic Shimura curves $\mathrm{Sh}_n(K_p)$ can be uniformized by $\mathbb{H}$ i.e. can be written as $\mathrm{Sh}_n(K_p) \cong \coprod_i \Gamma_i \backslash \Sigma^n$.

Then the sections of the push forward of $\mathscr{O}(k)$ on can be interpreted as "modular forms of weight k" on the associated Shimura curve (Shimura isomorphism).

Definition

Here we write X for $\mathbb{H}$, $\hat{\mathbb{H}}$ or $\overline{\mathbb{H}}$ and $H \subset G$ (most of the time G^+ or G).

Definition (H-equivariant line bundles)

An H-equivariant line bundle on X is a line bundle $\mathcal{L}$ with isomorphisms $(\rho_g : g^{-1}\mathcal{L} \xrightarrow{\sim} \mathcal{L})_{g \in H}$ satisfying

$$\forall g \in H, \forall f \in \mathscr{O}(U), \forall v \in \mathcal{L}(U), \rho_g(fv) = (g \cdot f)\rho_g(v) \quad \mathscr{O}[H] - \text{linear}$$

$$\forall g, h \in H, \rho_{gh} = \rho_g \circ g^{-1}\rho_h.$$

The group for the tensor product of H-equivariant line bundles up to strong equivalence is denoted by $\mathrm{Pic}_H(X)$.

Examples

Here are some examples of equivariant line bundles :

1. $\mathscr{O}(k)$ on $\mathbb{H}$ for G,
2. For any character ψ of H, we have $\mathscr{O}(\psi)$ an action of H on $\mathscr{O}$ given by
$$\rho_g(f) = \psi(g)g \cdot f,$$
3. on $\mathbb{H}$ for G^+ : $\pi_* \mathscr{O}_{\Sigma^n}^{\rho}$ and $\pi_*(\mathscr{O}_{\Sigma^n}(k))^{\rho}$ for ρ irreducible of $D^*/(1 + \mathfrak{m}_D^n)$ $\rightsquigarrow$ line bundles when $n = 1$,
4. $\Omega(\log)$ on $\hat{\mathbb{H}}$ for G an integral structure for $\mathscr{O}(-2)$.

Examples coming from the modular interpretation

Fundamental result : $\hat{\mathbb{H}}$ represents a problem of deformations of formal modules $\rightsquigarrow$ universal module $\mathfrak{X}$ on $\hat{\mathbb{H}}$. We can define more G^+-equivariant line bundles :

1. $\mathrm{Lie}(\mathfrak{X})$ admits an action of $\mathbb{Z}_{p^2}^* \subset \mathcal{O}_D^*$.
   Each isotypical part $\omega_i = (\mathrm{Lie}(\mathfrak{X})_i)^* \in \mathrm{Pic}_{G^+}(\hat{\mathbb{H}})$ and $\omega_i[1/p] \cong \mathscr{O}(-1)$,

2. the augmentation ideal $\mathcal{I}$ of the torsion points $\mathfrak{X}[\mathfrak{m}_D]$ admits an action of $(\mathcal{O}_D/\mathfrak{m}_D)^* \cong \mathbb{F}_{p^2}^*$ and each isotypical part

$$\mathcal{I} = \bigoplus_{\chi} \mathscr{L}_\chi$$

   is in $\mathrm{Pic}_{G^+}(\hat{\mathbb{H}})$ and $\mathscr{L}_\chi[1/p] \cong \mathscr{O}_{\Sigma_1}^{\chi}$. We distinguish two of them $\mathscr{L}_0$, $\mathscr{L}_1$ corresponding to the additve mod p characters of $\mathbb{F}_{p^2}^*$.

Classification

Definition

A G^+-equivariant line bundle on $\hat{\mathbb{H}}$ is said to be modular if it is a tensor product of $\mathscr{O}(\psi)$, ω_0, ω_1, $\mathscr{L}_0$, $\mathscr{L}_1$. The set of modular objects will be denoted $\mathrm{Pic}_{(mod)}(\hat{\mathbb{H}})$.

The classification result (the concise version) reads as follows :

Theorem (J.)

Any G^+-equivariant line bundle on $\hat{\mathbb{H}}$ is modular i.e. $\mathrm{Pic}_{G^+}(\hat{\mathbb{H}}) = \mathrm{Pic}_{(mod)}(\hat{\mathbb{H}})$.

Moreover the group $\mathrm{Pic}_G(\hat{\mathbb{H}})$ is generated by the characters and by $\Omega^1(\log)$

In particular, $\mathscr{O}(1)$ has no G-equivariant integral structure !

A useful exact sequence

For a general space $H \curvearrowright X$, the main tool to study $\mathrm{Pic}_H(X)$ is the following :

Propostition

We have an exact sequence :

$$0 \to \mathrm{H}^1(H, \mathscr{O}(X)^*) \to \mathrm{Pic}_H(X) \to \mathrm{Pic}(X)^H \to \mathrm{H}^2(H, \mathscr{O}(X)^*)$$

Example in generic fiber

On the generic fiber $\mathbb{H}$, things can be made a little bit simpler (not substantially...) by the following result

Theorem (J., ...)

$$\mathrm{Pic}(\mathbb{H}) = 0$$

Corollary

$$\mathrm{Pic}_H(\mathbb{H}) \cong \mathrm{H}^1(H, \mathscr{O}(\mathbb{H})^*)$$

Application on the integral level

As we have $\mathscr{O}(\hat{\mathbb{H}}) = \mathcal{O}_{\breve{K}}$, $\mathrm{Pic}(\hat{\mathbb{H}})^{G^+} \cong \mathbb{Z}^2$, $\mathrm{Pic}(\hat{\mathbb{H}})^{G} \cong \mathbb{Z}$ (see next slides), we have obtained the following sequence

Corollary

$$0 \to \hom(G^+, \mathcal{O}_{\breve{K}}^*) \to \mathrm{Pic}_{G^+}(\hat{\mathbb{H}}) \xrightarrow{(\mathrm{ord}_{s_0}, \mathrm{ord}_{s_1})} \mathbb{Z}^2$$
$$0 \to \hom(G, \mathcal{O}_{\breve{K}}^*) \to \mathrm{Pic}_{G}(\hat{\mathbb{H}}) \xrightarrow{\mathrm{ord}_{s_1}} \mathbb{Z}$$

Construction

Denote $\mathcal{BT}$ the Bruhat-Tits building :

- with vertices $\mathcal{BT}_0 = \mathrm{GL}_2(K)/K^* \, \mathrm{GL}_2(\mathcal{O}_K)$ the set of lattices of K^2 up to homothetie,
- with edges the set $\mathcal{BT}_1$ of $(s_0, s_1) = ([M_0], [M_1]) \in \mathcal{BT}_0^2$ satisfying

$$M_0 \supsetneq M_1 \supsetneq \varpi M_0.$$

Action of G^+ and G on the building

1. G acts transitively on $\mathcal{BT}_0$ but G^+ has two orbits $G^+ s_0$, $G^+ s_1$ ($s \in G^+ s_i$ iff $d(s, s_i)$ is even) with stabiliser $\mathrm{GL}_2(\mathcal{O}_K)\varpi^{\mathbb{Z}}$ up to conjugation,
2. G and G^+ act transitively on $\mathcal{BT}_1$. with stabiliser $I\varpi^{\mathbb{Z}}$ up to conjugation (I for Iwahori).

Description of the special fiber

- The irreducible components of the special fiber $\overline{\mathbb{H}}$ are in bijection with $\mathcal{BT}_0$ and are isomorphic to the projective line $(\mathbb{P}_s)_{s\in\mathcal{BT}_0}$,
- the intersections $\{(s,s') : \mathbb{P}_s \cap \mathbb{P}_{s'} \neq 0\}$ are in bijection with $\mathcal{BT}_1$ and the components meet transversaly on some closed points,
- for any fixed vertex s, the set $\{s' : \mathbb{P}_s \cap \mathbb{P}_{s'} \neq 0\}$ are in bijection with $\mathbb{P}_s(\mathbb{F})$.

In other words, $\mathcal{BT}$ is the nerve of the cover of $\overline{\mathbb{H}}$ by its irreducible components.

Bruhat-Tits building

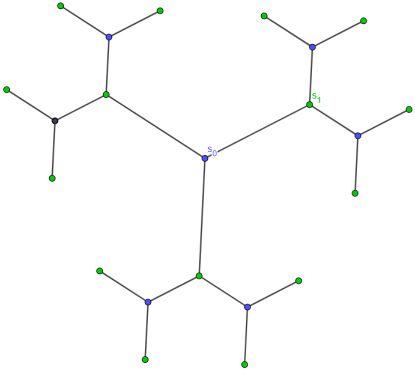


Figure: $\mathcal{BT}$ for $\mathrm{SL}_2(\mathbb{Q}_2)$

Components of $\overline{\mathbb{H}}$

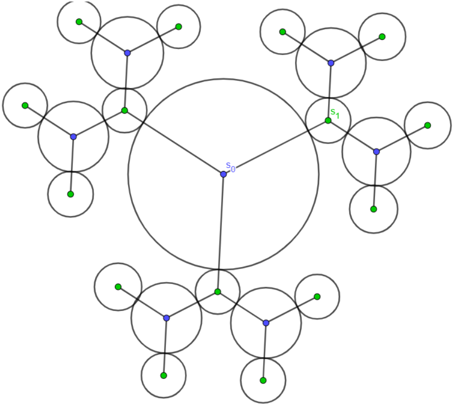


Figure: $\mathcal{BT}$ and $\overline{\mathbb{H}}$

$\mathrm{Pic}(\hat{\mathbb{H}})$

Theorem (J.)

$$\mathrm{Pic}(\hat{\mathbb{H}}) \xrightarrow{\sim} \mathrm{Pic}(\overline{\mathbb{H}}) \xrightarrow{\sim} \prod_{s\in\mathcal{BT}_0} \mathrm{Pic}(\mathbb{P}_s) \cong \prod_{s\in\mathcal{BT}_0} \mathbb{Z}$$

The first isomorphism is also true for most open of $\hat{\mathbb{H}}$. To see that it globalizes, we observe

$$\mathrm{H}^1_{\mathrm{an}}(\mathbb{H}, 1+\mathscr{O}^{++}) = 0.$$

Orders of modular bundles

To prove the classification result, we need to determine the image of $(\mathrm{ord}_{s_0}, \mathrm{ord}_{s_1})$. We begin by modular bundles :

Proposition

1. $\mathrm{ord}_{s_0}(\omega_0) = -1$ and $\mathrm{ord}_{s_1}(\omega_0) = q$.
2. $\mathrm{ord}_{s_0}(\omega_1) = q$ and $\mathrm{ord}_{s_1}(\omega_1) = -1$.
3. $\mathrm{ord}_{s_0}(\mathscr{L}_0) = 1$ and $\mathrm{ord}_{s_1}(\mathscr{L}_0) = -1$
4. $\mathrm{ord}_{s_0}(\mathscr{L}_1) = -1$ and $\mathrm{ord}_{s_1}(\mathscr{L}_1) = 1$
5. $\mathrm{ord}_{s_0}(\Omega^1(\log)) = \mathrm{ord}_{s_1}(\Omega^1(\log)) = q - 1$

In particular, the image of $\mathrm{Pic}_{(mod)}(\hat{\mathbb{H}})$ by $(\mathrm{ord}_{s_0}, \mathrm{ord}_{s_1})$ in $\mathbb{Z}^2$ is

$$\{(n, m) : n + m \equiv 0 \pmod{q-1}\}$$

Orders of modular bundles : Some arguments

For any non-zero map on the projective line $f : \mathscr{O}(k) \to \mathscr{O}(k')$, we have

$$k' - k = \sum_{y \in V^+(f)} \mathrm{mult}_y(f).$$

Idea : use this principle for maps from the modular problem.

This relies on making explicit these identifications coming from the representability

$\overline{\mathbb{H}}(\overline{\mathbb{F}}) \xrightarrow{\sim} \{\text{Formal module over } \overline{\mathbb{F}} + \ldots\} \xrightarrow{\sim} \{\text{Free } \mathcal{O}_{\breve{K}}\text{-module} + \Pi, V, F \text{ operator} + \ldots\}$.

Need also $\overline{\mathbb{H}}(\overline{\mathbb{F}}[\varepsilon])$ (Dieudonné-Messing theory)

Last step of the argument (finally!)

Proposition

For G^+-equivariant line bundles $\mathcal{L}$ on $\hat{\mathbb{H}}$ or $\overline{\mathbb{H}}$, we have :

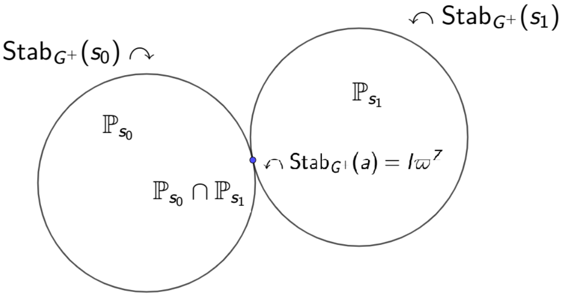
$$\mathrm{ord}_{s_0}(\mathcal{L}) + \mathrm{ord}_{s_1}(\mathcal{L}) \equiv 0 \pmod{q-1}$$

and for G-equivariant ones :

$$\mathrm{ord}_{s_0}(\mathcal{L}) \equiv \mathrm{ord}_{s_1}(\mathcal{L}) \equiv 0 \pmod{q-1}$$

In particular, the image of $\mathrm{Pic}_{(mod)}(\hat{\mathbb{H}})$ and $\mathrm{Pic}_{G^+}(\hat{\mathbb{H}})$ by $(\mathrm{ord}_{s_0}, \mathrm{ord}_{s_1})$ are the same.

The end of the proof



1. ord $s_i(\mathcal{L})$ determines the action of $\mathrm{Stab}(s_i)$ on $\mathcal{L}(\mathbb{P}_{s_i})$ (up to a character) and of $\mathrm{Stab}(a)$ on $\mathcal{L}(\mathbb{P}_{s_0} \cap \mathbb{P}_{s_1})$ which should be independent of i,
2. $\mathcal{L}(\mathbb{P}_{s_0} \cap \mathbb{P}_{s_1}) \cong_{\mathrm{Stab}(a)} \chi : \mathrm{Diag}(a, d) \mapsto a^{\mu_1} d^{\mu_2} \in (\mathbb{F}_q^*)^2$

Classification on the special fiber

As the above arguments work even on the special fiber, we have

Corollary

For $H = G^+$ or G, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{Gr}}(H, 1 + \varpi \mathcal{O}_{\breve{K}}) \to \mathrm{Pic}_H(\hat{\mathbb{H}}) \to \mathrm{Pic}_H(\overline{\mathbb{H}}) \to 0$$

More on the classification on the integral level

Fix $\mathcal{L} \in \mathrm{Pic}_{G^+}(\hat{\mathbb{H}})$ once and for all

Corollary

$\mathcal{L}$ can be written uniquely of the form

$$\mathcal{L} = \omega_0^{-w(\mathcal{L})} \otimes \mathscr{L}_0^{t(\mathcal{L})} \otimes \psi$$

where $w(\mathcal{L}) = -\frac{1}{q-1}(\mathrm{ord}_{s_0}(\mathcal{L}) + \mathrm{ord}_{s_1}(\mathcal{L}))$ is the weight and $t(\mathcal{L})$ is the type.

Classification on the generic fiber

As we have $\omega_0[1/p] \cong \mathscr{O}(-1)$, $\mathscr{L}_0[1/p] \cong \mathscr{O}_{\Sigma_1}^{\chi_0}$, we have on the generic fiber :

Corollary

$$\mathcal{L}[1/p] \cong (\mathscr{O}_{\Sigma_1}(w(\mathcal{L})))^{\chi_0^{t(\mathcal{L})}} \otimes \psi$$

Moreover, we have $\mathcal{L}_0[1/p] \cong \mathcal{L}_1[1/p]$ if and only if

$$w(\mathcal{L}_0) = w(\mathcal{L}_1) \text{ and } t(\mathcal{L}_0) \equiv t(\mathcal{L}_1) \pmod{q+1}.$$

Vanishing results

We have by studying this exact sequence

$$0 \to \mathcal{L} \xrightarrow{\times \varpi} \mathcal{L} \to \overline{\mathcal{L}} \to 0$$

Lemma

For $\mathcal{L}$ a G^+-equivariant line bundle, if $\mathrm{H}^i(\overline{\mathbb{H}}, \overline{\mathcal{L}}) = 0$ for some $i = 0, 1$, then

1. $\mathrm{H}^i(\hat{\mathbb{H}}, \mathcal{L}) = 0$,
2. $\mathrm{H}^{1-i}(\hat{\mathbb{H}}, \mathcal{L})$ is $\mathcal{O}_K$-flat
3. $\mathrm{H}^{1-i}(\overline{\mathbb{H}}, \overline{\mathcal{L}}) = \mathrm{H}^{1-i}(\hat{\mathbb{H}}, \mathcal{L})/\varpi$.

Application

As an application

Corollary

1. if $\mathcal{L}$ is positive (i.e. both orders are positive), then $\mathrm{H}^1(\overline{\mathbb{H}}, \overline{\mathcal{L}}) = 0$,
2. if $\mathcal{L}$ is negative (i.e. both orders are negative), then $\mathscr{L}(\overline{\mathcal{L}}) = 0$.

Cech argument

We can easily justify that we have a long exact sequence

$$0 \to \overline{\mathcal{L}}(\overline{\mathbb{H}}) \to \prod_{s \in \mathcal{BT}_0} \overline{\mathcal{L}}(\mathbb{P}_s) \to \prod_{a=(s,s') \in \mathcal{BT}_1} \overline{\mathcal{L}}(\mathbb{P}_s \cap \mathbb{P}_{s'})$$
$$\to \mathrm{H}^1(\overline{\mathbb{H}}, \overline{\mathcal{L}}) \to \prod_{s \in \mathcal{BT}_0} \mathrm{H}^1(\mathbb{P}_s, \overline{\mathcal{L}}) \to 0.$$

The different terms above can be written as an induction of characters of I or of representations $\sigma_{s,k} := \det^s \otimes \mathrm{Sym}^k \overline{\mathbb{F}}^2$.

Study of the Cech complexe

Corollary

The representation $\overline{\mathcal{L}}(\overline{\mathbb{H}})^*$ can only be irreducible when $\mathcal{L}$ is positive of weight -1. In that case,

$$\overline{\mathcal{L}}(\overline{\mathbb{H}})^* \cong \frac{\text{c-ind}_{I\varpi^{\mathbb{Z}}}^{G^+} \mu}{\text{c-ind}_{G^\circ\varpi^{\mathbb{Z}}}^{G} \sigma_{s,k} + (\text{c-ind}_{G^\circ\varpi^{\mathbb{Z}}}^{G} \sigma_{s+k,p-1-k})^{w}}$$

with $w \notin G^+$.

The case of $G^\circ = \mathrm{GL}_2(\mathcal{O}_K)$

Here K is totally ramified over $\mathbb{Q}_p$,

Proposition

Every irreducible representations mod p of $\mathrm{GL}_2(\mathbb{F}_p)$ and of $G^\circ \varpi^{\mathbb{Z}}$ (smooth) can be written uniquely of the form $(0 \leq s, k \leq q-1)$

$$\sigma_{s,k} = \det^s \otimes \mathrm{Sym}^k \overline{\mathbb{F}}^2$$

We would like to understand $\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G} \sigma_{s,k}$.

Hecke algebra

The Hecke algebra is $\mathcal{H}(\sigma_{s,k}) = \mathrm{End}_G(\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^G \sigma_{s,k})$

Lemma

We have :

$$\mathcal{H}(\sigma_{s,k}) \cong \overline{\mathbb{F}}_p[T]$$

for a distinguished operator T.

Lemma

Every smooth irreducible representations mod p of G is a quotient of

$$\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^G \sigma_{s,k}/(T-\lambda)$$

Representations of G

Theorem (Bartel-Livne)

Smooth irreducible representations mod p of G can be separated in four disjoint families :

1. characters ($k = 0$ and $\lambda = \pm 1$),
2. the special series $\chi \otimes \mathrm{St} = \chi \otimes (\text{c-ind}_B^G \mathbb{1}_B)/\mathbb{1}_G$ ($k = p-1$ and $\lambda = \pm 1$),
3. the principal series $\text{c-ind}_B^G \mu_1 \otimes \mu_2$ with $\mu_1 \neq \mu_2$ ($(k, \lambda) \in \{0, p-1\} \times \{\pm 1\}$ and $\lambda \neq 0$),
4. supersingular representations ($\lambda = 0$).

Supersingular representations form the most mysterious family except when $K = \mathbb{Q}_p$:

Theorem (Breuil)

When $K = \mathbb{Q}_p$, the following representations are irreducible

$$\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^G \sigma_{s,k} / T$$

Representations of $S = \mathrm{SL}_2(K)$

Theorem (Abdellatif, Chen)

Smooth irreducible representations mod p of $S := \mathrm{SL}_2(K)$ can be separated in four disjoint families :

1. the trivial representation $\mathbb{1}_S$,
2. the special representation $\mathrm{Sp} = \mathrm{St}\,|_S$,
3. the principal series $\text{c-ind}_B^G \mu \otimes \mathbb{1}|_S$ with $\mu \neq \mathbb{1}$,
4. supersingular representations (quotients of the restriction of the supersingular representations of G).

Theorem (Abdellatif, Chen)

When $K = \mathbb{Q}_p$, supersingular representations of G decompose as the sum of two irreducible representations of $\mathrm{SL}_2(K)$:

$$\frac{\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G} \sigma_{s,k}}{T}|_S = \frac{\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G^+} \sigma_{s,k}}{T}|_S \oplus \frac{(\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G^+} \sigma_{s,k})^w}{T}|_S$$

Relations between representations of G and G^+

Proposition

Take two smooth irreducible representations σ of G and ρ of G^+ :

1. $\sigma|_{G^+}$ and $\mathrm{ind}_{G^+}^{G} \rho$ are either irreducible or the sum of two irreducible representations,
2. any irreducible representation of G or G^+ can be obtained via this process.

Representations of G^+

Corollary

Smooth irreducible representations mod p of G^+ can be separated in four disjoint families :

1. characters,
2. the special series $\chi \otimes \mathrm{St}\,|_{G^+}$,
3. the principal series $\text{c-ind}_B^G \mu_1 \otimes \mu_2|_{G^+}$ with $\mu_1 \neq \mu_2$,
4. supersingular representations (direct summand of the restriction of supersingular representations of G).

Corollary

When $K = \mathbb{Q}_p$, supersingular representations of G decompose :

$$\frac{\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G} \sigma_{s,k}}{T}\Big|_{G^+} = \frac{\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G^+} \sigma_{s,k}}{T} \oplus \frac{(\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G^+} \sigma_{s,k})^w}{T}$$

Interesting consequences

Here $K = \mathbb{Q}_p$:

Corollary

The restriction to $\mathrm{SL}_2(\mathbb{Q}_p)$ induces a bijective map

$$\mathrm{Irr}_{G^+}(\overline{\mathbb{F}}_p)/\{\mathit{characters}\} \xrightarrow{\sim} \mathrm{Irr}_{\mathrm{SL}_2(\mathbb{Q}_p)}(\overline{\mathbb{F}}).$$

Corollary

A G-representation is supersingular if and only if it is the induction of an irreducible representation of G^+ if and only if its restriction to G^+ decomposes.

Other description of supersingular representations

Theorem (Anandavardhanan-Borisagar)

We have

$$\frac{\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G} \sigma_{s,k}}{T} \cong \frac{\text{c-ind}_{I\varpi^{\mathbb{Z}}}^{G} \mu}{\text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G} \sigma_{s,k} + \text{c-ind}_{G^\circ \varpi^{\mathbb{Z}}}^{G} \sigma_{s+k,p-1-k}}$$

This isomorphism is compatible with the previous decomposition on G^+ which proves the main result.