Equivariant line bundles on the Drinfeld Tower and p-adic local Langlands correspondence (part 1)

The sheaf of differential forms (by Dospinescu-Le Bras)

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Poincaré and Drinfeld half-plane

On $\mathbb{C}$, we have $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} = (\mathbb{P}^1(\mathbb{C}) \backslash \mathbb{P}^1(\mathbb{R}))^+$

Similarly, over $\mathbb{C}_p$, we have a Stein rigid open of $\mathbb{P}^1$:

$$\mathbb{H}(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(K)$$

with $K/\mathbb{Q}_p$ finite

Uniformization

$GL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$ by homographies and for any torsionfree congruence subgroup $\Gamma \subset GL_2(\mathbb{Z})$, consider the Riemann surface $Y_\Gamma = \Gamma \backslash \mathcal{H}$.

If $\Gamma = \Gamma(M) = \ker(GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/M\mathbb{Z}))$, write $Y(M) = Y_\Gamma$
$\rightsquigarrow$ tower of covers $\{Y(p^n)\}_{n \geq 3}$ with p a prime number.

Mumford curves : $\Gamma \backslash \mathbb{H}$ with discrete cocompact $\Gamma \subset GL_2(K)$.

Simple connectedness

Theorem (Van Der Put)

The spaces $\mathbb{H}$ are simply connected for the analytic topology. Moreover, one has for any constant sheaf A

$$\mathrm{H}^q_{\mathrm{an}}(\mathbb{H}, A) = 0 \text{ if } q \geq 1$$

"Universal" cover

For the étale topology, $\mathbb{H}$ admits a $\mathrm{GL}_2(K)$-equivariant universal cover which is perfectoïd (Drinfeld, Gross-Hopkins, Faltings, Fargues, Scholze, Weinstein)

$$\begin{array}{ccc} & \mathcal{M}^{\infty} & \\ {\scriptstyle \mathrm{GL}_2(\mathcal{O}_K)} \swarrow & & \searrow {\scriptstyle \mathcal{O}_D^*} \\ \mathcal{M}^0_{LT} := \mathring{\mathbb{B}}^1_K \times \mathbb{Z} & & \mathcal{M}^0_{Dr} := \mathbb{H} \times \mathbb{Z} \\ \downarrow {\scriptstyle \pi_{LT}} & & \downarrow {\scriptstyle \pi_{Dr}} \\ \mathcal{P}_{LT} := \mathbb{P}^1_K & & \mathcal{P}_{Dr} := \mathbb{H} \end{array}$$

with D the division algebra over K of invariant $1/2$, $\mathcal{O}_D$ the maximal order.

The towers of covers

Let's build two towers of covers $(\mathcal{M}_{LT}^n)_n$, $(\mathcal{M}_{Dr}^n)_n$ with these definitions :

$$\mathcal{M}_{Dr}^n := \mathcal{M}^\infty/(1+\mathfrak{m}_D^n) := \Sigma^n \times \mathbb{Z}$$
$$\mathcal{M}_{LT}^n := \mathcal{M}^\infty/(1+\mathrm{M}_2(\mathfrak{m}_K^n)) := \mathrm{LT}^n \times \mathbb{Z}$$

Geometric realizations

Each cover LT^n admits an action of $\mathrm{GL}_2(\mathcal{O}_K) \times D^* \times W_K$ and $\mathrm{GL}_2(K) \times D^* \times W_K$ for Σ^n.

Theorem (Drinfeld, Carayol, Harris-Taylor, Faltings ...)

$\varinjlim_n \mathrm{H}^*_{\text{ét},c}(\mathrm{LT}^n, \overline{\mathbb{Q}}_l)_{cusp}$ and $\varinjlim_n \mathrm{H}^*_{\text{ét},c}(\Sigma^n, \overline{\mathbb{Q}}_l)_{cusp}$ ($l \neq p$) provides geometric realizations of local Langlands and Jacquet-Langlands' correspondences.

Realizations : The statement

Theorem

Let π be a supercuspidal representation of G, with $\rho := \mathrm{JL}(\pi)$ factoring through $D^*/(1+\mathfrak{m}_D^n)$:

$$\mathrm{H}^i_{\text{ét},c}(\Sigma^n, \overline{\mathbb{Q}}_l)^\rho \cong_{G\times W_K} \begin{cases} \pi \otimes \mathrm{LL}(\pi) & \text{If } i = 1 \\ 0 & \text{If not} \end{cases}$$

with $\mathrm{H}^i_{\text{ét},c}(\Sigma^n, \overline{\mathbb{Q}}_l)^\rho := \mathrm{Hom}(\rho, \mathrm{H}^i_{\text{ét},c}(\Sigma^n, \overline{\mathbb{Q}}_l))$.

Classification of smooth irreducible G-representations

Set $C \cong \hat{\overline{\mathbb{Q}}}_l$, we will study smooth irreducible $G = \mathrm{GL}_2(\mathbb{Q}_p)$-representations (π, V) (Stab(v) is open in G for any $v \in V$).

They can be distinguished into four families :

1. smooth characters of G,
2. the special series $\chi \otimes \mathrm{St} = \chi \otimes (\mathrm{Ind}_B^G \mathbb{1}_B)/\mathbb{1}_G$,
3. the principal series $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ with $\mu_1/\mu_2 \neq |\cdot|^{\pm 1}$,
4. supercuspidal representations.

Classical correspondences

Theorem (Classical local Jacquet-Langlands' and Langlands' correspondences

There exists a natural bijection $\pi \mapsto \mathrm{LL}(\pi)$ between isomorphism classes

1. Supercuspidal representations of G on C,
2. Irreducible 2-dimensionnal representations of W_K on C.

There exists a natural bijection $\pi \mapsto \mathrm{JL}(\pi)$ between isomorphism classes

1. Supercuspidal representations of G on C,
2. Smooth irreducible representations of dimension > 1 of D^* on C.

We will fix once and for all π a supercuspidal representation of G, with $\rho := \mathrm{JL}(\pi)$ factoring through $D^*/(1 + \mathfrak{m}_D^n)$

Motivation

Conjecture

$\varinjlim_n \mathrm{H}^*_{\mathrm{dR},c}(\mathrm{LT}^n)_{cusp}$ and $\varinjlim_n \mathrm{H}^*_{\mathrm{dR},c}(\Sigma^n)_{cusp}$ also provides geometric realizations of local Langlands and Jacquet-Langlands' correspondences.

Correspondence for De Rham cohomology

Theorem (Dospinescu-Le Bras)

$$\mathrm{H}^1_{\mathrm{dR},c}(\Sigma^n)^\rho \cong_G \pi \otimes M_{Dr}(\pi)$$

with $\dim_L M_{Dr}(\pi) = 2$.

- Drinfeld and Lubin-Tate side when $d = 1$, for any K : Colmez-Dospinescu-Niziol
- Drinfeld side when $n = d = 1$, $K = \mathbb{Q}_p$: Breuil-Strauch, Lue Pan (+ semi-stable model)
- Drinfeld and Lubin-Tate side when $n = 1$, for any K and dimension : J. (+ equation) building upon Wang's and Yoshida's l-adic results.

Construction of $(\Omega^1)^{\rho}_{\mathcal{L}}$

Given a line $\mathcal{L} \subset M_{dR}(\pi)$, we define

$$(\Omega^1)^{\rho}_{\mathcal{L}} \subset \Omega^1(\Sigma^n)^{\rho}$$

the pre-image of $\mathcal{L}^{\perp} \otimes \pi^* \subset \mathrm{H}^1_{\mathrm{dR}}(\Sigma^n)^{\rho}$.

By definition, it sits in an exact sequence

$$0 \to \mathscr{O}(\Sigma^n)^{\rho} \to (\Omega^1)^{\rho}_{\mathcal{L}} \to \pi^* \to 0$$

which dually gives

$$0 \to \pi \to ((\Omega^1)^{\rho}_{\mathcal{L}})^* \to (\mathscr{O}(\Sigma^n)^{\rho})^* \to 0$$

Main Goal

Motivation

Some aspects of the p-adic local Langlands correspondence should be visible in the cohomology of vector bundles on the coverings Σ^n.

We will illustrate this principle :

1. today, by studying the structure sheaves $\mathscr{O}_{\Sigma^n}$ and the differential forms $\Omega^1_{\Sigma^n} \cong \mathscr{O}_{\Sigma^n}(-2)$,
2. next time, by looking at integral structures of $\mathscr{O}_{\Sigma^1}(-1)$ and their associated representations in the mod p setting.

Framework

Let $L/\mathbb{Q}_p$ finite, we will study the following kind of representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ on L-vector spaces

1. Representations on L-Banach spaces which are unitary (with a G-lattice Θ) and admissible ($(\Theta/\varpi_L)^H$ is finite dimensional for any open $H \subset G$),
2. Locally analytic representations on compact type L-spaces.

To a L-Banach space representation Π, we can associate a smooth one Π^{lisse} (which is trivial most of the time) and a locally analytic one Π^{an} (which is dense in Π).

p-adic local Langlands : The Banach space side

An irreducible Banach space representation of G is supersingular if it is not a quotient of the induction of a character of the Borel.

Theorem (Colmez, Paskunas, Dospinescu...)

We have bijections Π, V inverse to each other

$$\{\text{supersingular representations}\} \underset{\Pi}{\overset{V}{\rightleftharpoons}} \{\rho : \mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(L) \text{ continous irreducible}\}$$

The set $\mathcal{V}(\pi)$

Recall we have fixed π smooth supercuspidal, consider

$$\mathcal{V}(\pi) := \{\Pi : \Pi^{lisse} \cong \pi\}$$

Theorem (Colmez, Emerton)

For $\Pi \in \mathcal{V}(\pi)$, $V(\Pi)$ is de Rham with Hodge-Tate weights $(0,1)$ and

$$\mathcal{V}(\pi) \cong \mathbb{P}(M_{dR}(\pi))$$

The structure of $\mathcal{V}(\pi)$

To explain the interpretation of $\mathcal{V}(\pi)$ in terms of the correspondence, we need the following result :

Theorem (Colmez-Fontaine)

We have an equivalence of categories $V \mapsto D_{pst}(V)$ between

1. de Rham representations,
2. and weakly admissible (technical condition) filtered $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-modules.

For $\Pi \in \mathcal{V}(\pi)$ corresponding to $\mathcal{L} \subset M_{dR}(\pi)$,

1. $\pi \rightsquigarrow (\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-module on $D_{pst}(V(\Pi))$,
2. $\mathcal{L} \rightsquigarrow$ Hodge filtration on $D_{pst}(V(\Pi))$,

Locally analytic vectors

Given a line $\mathcal{L} \subset M_{dR}(\pi)$ corresponding to $\Pi_{\mathcal{L}} \in \mathcal{V}(\pi)$, we have naturally

$$0 \to \pi \to \Pi_{\mathcal{L}}^{an} \to \Pi_{\mathcal{L}}^{an}/\pi \to 0$$

where $\Pi_{\mathcal{L}}^{an}/\pi$ is absolutely irreducible by a result of Colmez.

Recall also that we have an exact sequence (with $\rho = \mathrm{JL}(\pi)$)

$$0 \to \pi \to ((\Omega^1)^{\rho}_{\mathcal{L}})^* \to (\mathscr{O}(\Sigma^n)^{\rho})^* \to 0$$

$\mathcal{V}(\pi)$ in the De Rham complex of Σ^n

Theorem (Dospinescu-Le Bras)

We have a commutative diagram with exact horizontal lines and isomorphic vertical maps :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi & \longrightarrow & ((\Omega^1)^{\rho}_{\mathcal{L}})^* & \longrightarrow & (\mathscr{O}(\Sigma^n)^{\rho})^* & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & \pi & \longrightarrow & \Pi^{an}_{\mathcal{L}} & \longrightarrow & \Pi^{an}_{\mathcal{L}}/\pi & \longrightarrow & 0 \end{array}$$

Some integral results in level 1

For $n = 1$, we even have a semi-stable model for Σ^1 and then an integral structure $\Omega^1(\log)$ for $\Omega^1_{\Sigma^1}$.

With similar constructions on $\Omega^1(\log)$, Lue Pan constructs a unitary Banach space-representation $B(\pi, \mathcal{L})$ conjectured to be $\Pi_{\mathcal{L}}$.

A global curve

Consider a quaternion algebra $B/\mathbb{Q}$ split at infinity and ramified at p (i.e. $B(\mathbb{R}) \cong \mathrm{M}_2(\mathbb{R})$, $B(\mathbb{Q}_p) \cong D$).

For any sufficiently small compact open subgroup $K_f \subset B^*(\mathbb{A}_f)$, we can define a smooth proper curve Sh_{K_f} over $\mathbb{Q}$ with $\mathbb{C}$-points :

$$\mathrm{Sh}_{K_f}(\mathbb{C}) = B^*(\mathbb{Q})\backslash((\mathbb{C}\backslash\mathbb{R}) \times B^*(\mathbb{A}_f)/K_f)$$

Global uniformization

We have a finite set :

$$X(K_f) = B^*(\mathbb{Q})\backslash B^*(\mathbb{A}_f)/K_f \cong \coprod_i \Gamma_i\backslash \mathrm{GL}_2(\mathbb{R})$$

which gives the following description of the Shimura curve

$$\mathrm{Sh}_{K_f} \cong \coprod_i \Gamma_i\backslash(\mathbb{C}\backslash\mathbb{R})$$

Local curve

For a local version, take an integer n and a subgroup K_f of the form

$$K_f = \mathcal{O}_D^* K^p \ (n = 0), \quad K_f = (1 + \mathfrak{m}_D^n) K^p \ (n \geq 1)$$

with $K^p \subset B^*(\mathbb{A}_f^p)$ sufficiently small.

We can define a local Shimura curve $\mathrm{Sh}_n(K^p) = (\mathrm{Sh}_{K_f} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{an}$.

Cerednik-Drinfeld uniformization

Now, consider the quaternion algebra $\overline{B}$ ramified at infinity, split at p and with the same invariant as B at any other places, then $K^p \subset B^*(\mathbb{A}_f^p) = \overline{B}^*(\mathbb{A}_f^p)$.

Once again, write

$$X(K^p) = \overline{B}^*(\mathbb{Q})\backslash\overline{B}^*(\mathbb{A}_f^p)/K^p \cong \coprod_i \Gamma_i\backslash G$$

Theorem (Cerednik, Drinfeld)

We have :

$$\mathrm{Sh}_n(K^p) \cong \coprod_i \Gamma_i\backslash\Sigma^n$$

compatible with n, K^p and the action of the Hecke algebra T.

Consequences on differential forms

By the Hochschild-Serre spectral sequence (and Σ^n is Stein), we have :

$$\Omega^1(\mathrm{Sh}_n(K^p)) \cong \bigoplus_i (\Omega^1(\Sigma^n))^{\Gamma_i}$$

which gives

Lemma

$$\Omega^1(\mathrm{Sh}_n(K^p))^\rho \cong \mathrm{Hom}_G^{cont}((\Omega^1(\Sigma^n)^\rho)^*, \mathrm{LA}(X(K^p))$$

" Similarly "

$$\mathrm{H}^1_{\mathrm{dR}}(\mathrm{Sh}_n(K^p))^\rho \cong \mathrm{Hom}_G^{cont}((\mathrm{H}^1_{\mathrm{dR},c}(\Sigma^n)^{\rho^*}, \mathrm{LC}(X(K^p))$$

Local-global compatibility

Given a prime ideal $\mathfrak{p}$ on T, we can describe :

1. $\mathrm{LC}(X(K^p))[\mathfrak{p}]$ and $\mathrm{LA}(X(K^p))[\mathfrak{p}]$ via the local-global compatibility à la Emerton,
2. $\Omega^1(\mathrm{Sh}_n(K^p))^\rho[\mathfrak{p}]$ in terms of automorphic representations.

In particular, we can choose $\mathfrak{p}$ so that

$$\mathrm{LA}(X(K^p))[\mathfrak{p}] \cong (\Pi^{an}_{\mathcal{L}})^r \quad (r > 0), \quad \Omega^1(\mathrm{Sh}_n(K^p))^\rho[\mathfrak{p}] \neq 0$$

Construction of $\alpha : (\Pi_{\mathcal{L}}^{an}/\Pi_{\mathcal{L}}^{lisse})^* \to \mathscr{O}(\Sigma^n)$

As an application,

Corollary

There is a nonzero map $\alpha : (\Pi_{\mathcal{L}}^{an})^* \to \Omega(\Sigma^n)$ and it induces another nonzero map (look at the smooth vectors of the duals)

$$\alpha : (\Pi_{\mathcal{L}}^{an}/\Pi_{\mathcal{L}}^{lisse})^* \to \mathscr{O}(\Sigma^n).$$

A similar argument shows that

Corollary

$$\dim_L \mathrm{Hom}_G(\pi, \mathrm{H}^1_{\mathrm{dR},c}(\Sigma^n)^\rho) = 2$$

α is injective (irreducibility of $\Pi_{\mathcal{L}}^{an}/\Pi_{\mathcal{L}}^{lisse}$) and it remains to show that it is surjective.

The operator ∂

This crucial step is one of the most technical points of the paper.

Theorem

There exists a structure of $\mathscr{O}(\mathbb{H})$-module on $(\Pi_{\mathcal{L}}^{an}/\Pi_{\mathcal{L}}^{lisse})^*$ which makes α linear.

A key step is to define the operator of multiplication by the variable $z \in \mathscr{O}(\mathbb{H})$ via the action of $\mathfrak{g} = \mathrm{Lie}(G) \cong \mathrm{M}_2(\mathbb{Q}_p)$ on $(\Pi_{\mathcal{L}}^{an}/\Pi_{\mathcal{L}}^{lisse})^*$.

Lemma

There exists a unique operator on $(\Pi_{\mathcal{L}}^{an}/\Pi_{\mathcal{L}}^{lisse})^*$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \mathrm{Id} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \circ \partial.$$

Morita duality

To show that ∂ the right operator, we need the following description of $\mathscr{O}(\mathbb{H})$:

Theorem (Morita)

Write St^{an} for $\mathrm{LA}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{1}$, we have a G-equivariant isomorphism :

$$\mu \in (\mathrm{St}^{an})^* \mapsto f_\mu(z) := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-x}\mu(x)dz \in \Omega^1(\mathbb{H})$$

We first show that we can define the action of $\frac{1}{z-x}$ for $x \in \mathbb{Q}_p$ via ∂ and then that the right integrals converge.

A topological lemma

The rest of the talk will try to explain this point.

Lemma

The map α has dense image.

The result will then follow this general topological statement :

Lemma

Consider the following data

1. a Stein rigid variety X,
2. a Frechet $\mathscr{O}(X)$-module M,
3. a finite projective $\mathscr{O}(X)$-module N,

then any continous $\mathscr{O}(X)$-linear map with dense image $M \to N$ is surjective.

Reduction to vector bundles

Consider :

$$0 \to \overline{\mathrm{Im}(\alpha)} \to \mathscr{O}(\Sigma^n)^\rho \to \mathscr{O}(\Sigma^n)^\rho / \overline{\mathrm{Im}(\alpha)} \to 0$$

It yields an exact sequence of coherent sheaves ($\mathbb{H}$ is Stein)

$$0 \to \mathscr{F} \to \mathscr{O}_{\Sigma^n}^\rho \to \mathscr{F}' \to 0$$

If $\mathscr{O}(\Sigma^n)^\rho / \overline{\mathrm{Im}(\alpha)} \neq 0$, their torsion is then trivial, they are all vector bundles.

Transfer to the Lubin-Tate side

Recall that we have a G-torsor $\pi_{LT} : \mathcal{M}_\infty \to \mathcal{P}_{LT} \cong \mathbb{P}^1$ and a D^*-torsor $\pi_{Dr} : \mathcal{M}_\infty \to \mathcal{P}_{Dr} \cong \mathbb{H}$.

As $\mathscr{O}^{\rho}_{\Sigma^n} = (\rho^* \otimes \mathscr{O}_{\mathcal{M}_\infty})^{D^*} = (\pi_{Dr,*}\pi^*_{LT}(\rho^* \otimes \mathscr{O}_{\mathbb{P}^1}))^{D^*}$, we have :

$$(\pi_{LT,*}\pi^*_{Dr}\mathscr{O}^{\rho}_{\Sigma^n})^G = (\pi_{LT,*}(\rho^* \otimes \mathscr{O}_{\mathcal{M}_\infty}))G = \rho^* \otimes \mathscr{O}_{\mathbb{P}^1}$$

It is sufficient to prove that $\rho^* \otimes \mathscr{O}_{\mathbb{P}^1}$ is irreducible in $\mathrm{Bun}_{D^*}(\mathbb{P}^1)$.